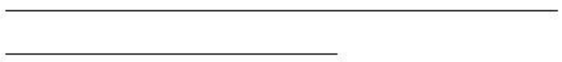
Was $\sqrt{5}$ the First Irrational Number?

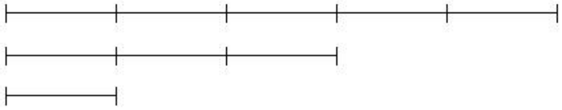
Since there are no surviving documents from this era to verify which was the first irrational number found we can not be sure exactly which one it was but there are some doubts that it was $\sqrt{2}$. The proof that $\sqrt{2}$ is irrational has a relatively high level of abstraction, even though it is easy for us to understand. Specifically, looking at a general rational number as a quotient of two integers that are represented as variables and the proof uses algebraic techniques that are also doubted to have existed at the time. To get a feel for how mathematics was done back then just pick up a translation to Euclid's Elements. The statements are entirely prose, there is no algebraic symbolism in the original text. For us, showing that $\sqrt{2}$ is irrational is an easy proof since we grew up with this level of abstraction but for someone living in 500 BC the proof would be all Greek to them.

Back then the terms rational and irrational really did not exist. They thought that every two magnitudes (or lengths) were commensurable. This meant that for every two given lengths there is a third length that divides the first two evenly. For example, take these two lengths

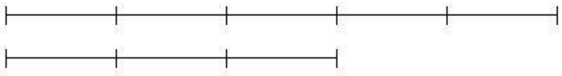


We can find a third length

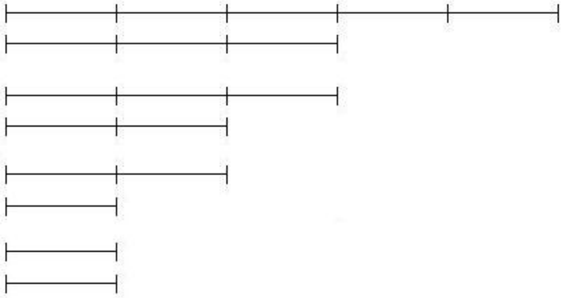
that divides both of the first two evenly.



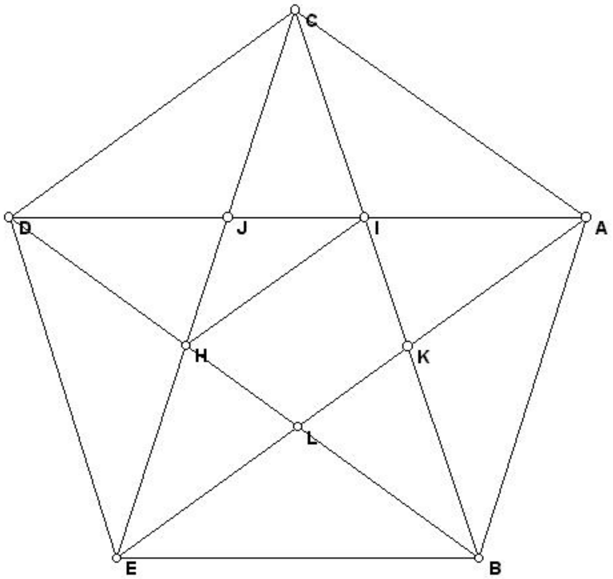
This is equivalent to saying that the ratio of the first two lengths is a rational number, in our current mathematical lingo. So the Pythagoreans, and all those before them thought that all numbers were rational. Another way to look at this concept of commensurability is to examine the relationship between the ratio of the original lengths a and b and the ratio of the lengths a and $b - a$, where a is the shortest length. Take our above example,



If we continually subtract the smaller length from the larger length and look at the ratios of the resulting lengths, eventually, the lengths are the same and hence the ratio is 1. In fact, when the lengths are the same it is the length that divides the first lengths evenly. Isn't that neat?



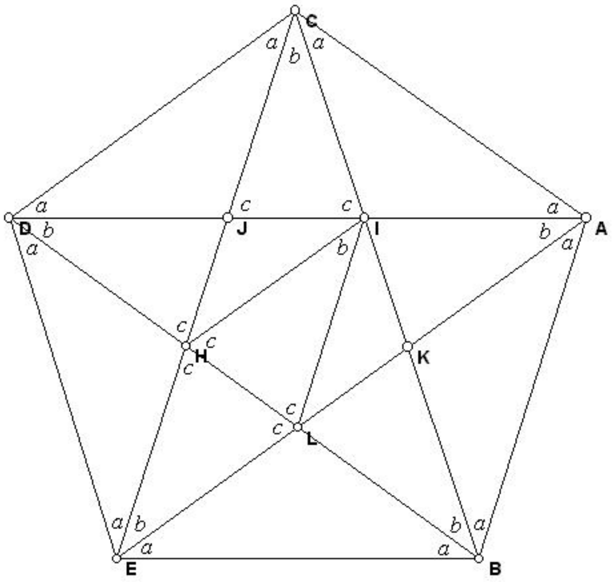
Now that have an idea of what they thought they could do with all lengths, let's see what happened when they were playing around with a pentagon. An interesting fact about pentagons is that if you draw the diagonals (that is join every second points) then a smaller pentagon will be formed in the center that is similar to the one you started with. That is, the scaling of the sides is the same for all sides. We will use a regular pentagon for this discussion, all the sides are equal in length. Not that it matters here, but we will say that the sides of the larger pentagon are all 1 unit in length.



What we are concerned with is the ratio of the lengths CE and CD. That is, the number $\frac{CE}{CD}$. Since $CD = 1$ we are really looking at the length CE. We need to show several relationships between lengths on this picture, most are verified through isosceles triangles. First note that ΔDCA is isosceles since $DC = CA = 1$. So $\angle CDA = \angle CAD = a$. We will also call $\angle ECB = b$. By symmetry we have $b = \angle ECB = \angle BDA = \angle AEC = \angle CBD = \angle EAD$. Since ΔDBC is isosceles $\angle BDC = \angle BCD$ and hence $\angle JCD = a$. This gives ΔDJC isosceles. By similarity we also have ΔCIA, ΔAKB, ΔBLE and ΔEHD isosceles.

If we look at ΔCEB we have we have $3b+2a = 180$. In ΔCDH we have $b+2a+\angle DHC = 180$, so $\angle DHC = 2b$. Also, ΔJCI is isosceles since $CJ = JD$ and by symmetry $JD = CI$.

Similarly, the other four "points" of the internal star are isosceles triangles. This gives $\angle CJI = \angle CIJ = c$. So from ΔCJI, $b + 2c = 180$ and hence $3b + 2a = b + 2c$, thus $c = a + b$. This gives $a + b = 2b$ and hence $a = b$, so ΔCDH is isosceles. Now before we get to the punch-line we need to consider the quadrilateral $EHIL$. We know that ΔEHL is isosceles with $\angle HEL = b$ and $\angle EHL = \angle ELH = c = 2b$. Also ΔIHL is isosceles since $IH = IL$ are both diagonals of the smaller regular pentagon. Furthermore, $\angle HIL = b$ since $\angle JIK = \angle DCA$ and the angle formed by the diagonals trisect the internal angle of the pentagon. This gives $\angle IHL = \angle ILH = c$. Hence quadrilateral $EHIL$ is a parallelogram. Furthermore, $EH = EL = HI$. An image with all of the angles labeled is below.



What are the important parts of what we did above? The main things to keep in mind are $CD = CH$, $EH = HI$, and $HI = CJ (= EH)$.

So if we start with the lengths CD and CE then $CD - CD = CE - CH = EH = HI$. Now we look at CD and HI, equivalently CH and HI. Doing the subtraction again gives $CH - HI = CH - CJ = HJ$. Now we are looking at the lengths HI and HJ.

This is equivalent to two steps on the length subtraction process. But the result is still the ratio of the side of a regular pentagon with a diagonal. So the ratio did not change in two steps of the process, and hence will never change. Thus, we will never get the ratio to be one with this subtraction process and as a result we will never be able to find that length that divides the first two evenly. Therefore we have found two lengths that are not commensurable.

It might surprise you to know that this is simpler than the $\sqrt{2}$ proof, at least for a mathematician in 500 BC. All of the argument was geometric, there was no part that was algebraic. We did use algebra to show that $a = b$, but this was just to make the argument a little easier to follow and certainly was not needed.

Let's find out what the length CE really is. Since we did two steps of the process we looked at the following sequence of ratios,

$$\frac{CE}{1} \longrightarrow \frac{1}{CE-1} \longrightarrow \frac{CE-1}{1-(CE-1)} = \frac{CE-1}{2-CE}$$

Since these are the same ratios we get the equation, replacing CE with x,

$$x = \frac{x-1}{2-x}$$

Solving this

$$\begin{aligned} x &= \frac{x-1}{2-x} \\ 2x - x^2 &= x - 1 \\ x^2 - x - 1 &= 0 \\ x &= \frac{1 \pm \sqrt{1-4(1)(-1)}}{2} \\ x &= \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

Since x is a length we must have $x = \frac{1+\sqrt{5}}{2}$, the golden ratio or golden mean.

It is argued that the first person to discover incommensurables was Hippasus. Since the Pythagoreans were a religious cult who literally worshiped numbers and felt that the act of commensuration lengths was perfection, the existence of incommensurables was akin to showing the Pythagoreans a picture of Satin. It is mythed that a small group of Pythagoreans tied Hippasus up, took him out to the middle of the Mediterranean Sea and tossed him in.